struction.

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METHOD FOR THE APPROXIMATE SOLUTION OF THE BELLMAN EQUATION FOR PROBLEMS OF OPTIMAL CONTROL OF SYSTEMS SUBJECT TO RANDOM PERTURBATIONS

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We propose a method for the approximate soluti n of the Bellman equation for problems of optimal control of the final state o a system containing Gaussian white noise of small intensity. We examine the case when the solutions of the deterministic Bellman equation, corresponding to a noisefree system, have discontinuities of the first kind in their own values or in the values of their derivatives. We have found the necessary and sufficient conditions for the synthesis of optimal control of a system additively containing Gaussian white noise to coincide with the corresponding synthesis for the deterministic problem. We prove estimates on the error in the method and we cite examples. Earlier the author had examined an analogous method for a restricted class of optimal control problems [1]. Certain methods for the approximate solution of the Bellman equation were studied in

[2−4] under the assumption that the deterministic Bellman equation has a sufficiently smooth solution. Certain exact solutions and estimates were obtained in [5].

1. Statement of the problem. The Bellman equation. Let the equation describing the motion of a system have the form

$$dx / dt = a(t)u + \varepsilon b(t)\xi \tag{1.1}$$

Here $t \in [0, T]$, x is an n-dimensional vector, u is a vector-valued control function of dimension m $(m \leqslant n)$ taking values in a closed convex m-dimensional set U, $u \in U$, ξ is an n-dimensional random perturbation vector, $a(t)$ is an $(n \times m)$-matrix, $b(t)$ is an $(n \times n)$-matrix nonsingular for all $t \in [0, T]$, ε is a small parameter. The elements of matrices a and b are assumed to be smooth functions of their arguments. As the random perturbation vector we consider a Gaussian white noise of unit intensity. Knowing the initial position $x(0) = x_0$, we are required to construct a control method which minimizes (or maximizes) the mean of some function

$$J = \psi[x(t)] \tag{1.2}$$

at the final instant $t = T$. It is assumed that the function $\psi(x)$ is bounded for all values of x.

Note. Problems in which the equation of motion is given as

$$dx/dt = A(t)x + a(t)u + \varepsilon b(t)\xi$$

reduce to problem (1.1) by a change of variables. Here $A(t)$ is an $(n \times n)$-matrix with coefficients depending smoothly on $t \in [0, T]$; $a(t)$, $b(t)$, u, ξ, ε have the same meaning as in Eq. (1.1).

The Bellman equation for problem (1.1) has the form [2−5]

$$S_\tau = \min_{u \in U} \left\{ \sum_{k=1}^{m} u_k \sum_{i=1}^{n} a_{ik} S_{x_i} \right\} + \frac{1}{2} \varepsilon^2 \operatorname{sp}(bb' S_{xx}) \tag{1.3}$$

$$S(x, 0) = \psi(x) \quad \left(\operatorname{sp}(bb' S_{xx}) = \sum_{i,k=1}^{n} b_{ik}^1 S_{x_i x_k} \right)$$

Here $S(x, \tau)$ is the Bellman function, $T - t = \tau$ is reverse time, the subscripts on function S denote the taking of the corresponding partial derivatives, a_{ik} $(i = 1, \ldots, n, k = 1, \ldots, m)$ are the elements of matrix a. Because matrix b is nonsingular,

$$\sum_{i,k=1}^{n} b_{ik}^1 \lambda_i \lambda_k > 0, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \neq 0, \ t \in [0, T]$$

Thus, the problem posed reduces to solving a Cauchy problem for the parabolic equation (1.3).

2. Regularization of the deterministic problem. Definition of the characteristic curves of the deterministic equation. We consider a deterministic problem (1.1), i.e. the problem without random perturbations,

$$dx/dt = a(t)u, \quad t \in [0, T], \quad u \in U \tag{2.1}$$

Here a is the same matrix as in (1.1). The functional to be minimized at the final instant has the form

$$J^\circ = \psi\,[x\,(T)] \tag{2.2}$$

We assume that the function S° corresponding to the deterministic problem (2.1), (2.2) is known and

$$S^\circ(x,\tau) = \begin{cases} S^{\circ 1}(x,\tau), & A^\circ(x,\tau) \leqslant 0 \\ S^{\circ 2}(x,\tau), & A^\circ(x,\tau) > 0 \end{cases}$$

Here $A^\circ(x,\tau)$ is some continuous function of its arguments. The functions $S^{\circ k}$ $(k = 1, 2)$, continuous and thrice differentiable outside set $A^\circ = 0$, are arranged so that the function S° has discontinuity of the first kind in its own values or in the values of its first two derivatives on the surface $A^\circ = 0$. It is clear that such a case corresponds to a discontinuous initial-value function $\psi(x)$. We assume further that the function S° is bounded for all values of x.

Let us formally write out the Bellman equation for the deterministic problem

$$S_\tau = \min_{u\in U}\left\{\sum_{k=1}^{m} u_k \sum_{i=1}^{n} a_{ik} S^\circ_{x_i}\right\}, \quad S^\circ(x,0) = \psi(x) \tag{2.3}$$

The boundary-value problem (2.3) with discontinuous initial data should be understood in the generalized sense [6].

For the subsequent constructions an essential role is played by the concept of characteristic curves of Eq. (2.3), which in the given case requires an additional clarification. From optimal control theory it is well known (see [7]) that under specific constraints on the smoothness of the functions S° the characteristics of the deterministic Bellman equation are the optimal trajectories of the motion of problem (2.1), (2.2). However, a direct application of the maximum principle to the deterministic problem (2.1) is impossible in view of the discontinuity of the functional (2.2) on the final value, which leads to ambiguity in the choice of the optimal control and of the optimal trajectories.

Example 1. Let us clarify what we have said by the example

$$dx/dt = u + \xi, \qquad |u| \leqslant k$$

where x is a scalar, ξ is Gaussian white noise of unit intensity. We are required to construct a control method which would maximize the probability of hitting onto the set $[-1, 1]$. The corresponding deterministic problem with $\xi = 0$ has the final-state functional

$$J^\circ = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases}$$

which we are required to maximize for $t = T$. We can convince ourselves that the region from which we can reach the segment $[-1, 1]$ at the instant T is given by the inequality $|x| \leqslant k\,(T - t) + 1$. Let us consider the optimal trajectories which pass through the point $t = t_0$, $x = x_0$ located inside the attainability region. It is clear that there can be many methods for hitting onto the segment $[-1,1]$ from the point (x_0, t_0). For example, if the point (x_0, t_0) lies inside the set $|x| \leqslant k\,(T - t)$, we can reach the set $x = 0$ at the instant $t_1 = t_0 - x_0 / k$ by setting $u = k$ and then to set $u = 0$. We can reach the set $x = k\,(T - t) + 1$ at the instant $t_2 = {}^1/_2 t_0 - (x_0 - 1) / 2k$ by setting $u = -k$ and next set $u = k$, etc. The control is not defined outside the attainability set since any possible control does not change the value of the final-value functional

(2.2).

In order to select uniquely the synthesis of the optimal control $u^\circ(x, t)$ and the corresponding field of optimal trajectories, and by the same token to determine the characteristic curves of the deterministic Bellman equation (2.3), we carry out the following additional constructions. Following the method presented in [1], we consider the function

$$S^\mu(x,\tau) = (2\sqrt{\pi})^{-n}(\mu\tau)^{-n/2}\int S^\circ(\lambda,\tau)\exp\left[-\frac{|\lambda-x|^2}{4\mu\tau}\right]d\lambda \qquad (2.4)$$

$$|\lambda - x| = \left[\sum_{j=1}^{n}(\lambda_j - x_j)^2\right]^{1/2}$$

Here μ is a positive number, $S^\circ(x, \tau)$ is the Bellman function of the deterministic problem; the integration is carried out over all the values, $-\infty \leqslant \lambda_i \leqslant +\infty$, $i = 1, \ldots, n$. From the properties of the fundamental solution of parabolic equations follow [8]: (a) $S^\mu(x, \tau)$ is an infinitely differentiable function of the x_i, $i = 1, \ldots, n$ for all $\mu > 0$, (b) $\lim\limits_{\mu\to 0} S^\mu(x, \tau) = S^\circ(x, \tau)$. For each $\mu \to 0$ we can find $u^\mu \in U$ on which

$$\min_{u\in U}\left\{\sum_{k=1}^{m} u_k \sum_{i=1}^{n} a_{ik} S^\mu_{x_i}\right\} = \sum_{k=1}^{m} u_k^\mu \sum_{i=1}^{n} a_{ik} S^\mu_{x_i} \qquad (2.5)$$

is reached. To u^μ constructed in this manner there corresponds a certain field of trajectories of problem (2.1), (2.2)

$$x = \zeta^\mu(t, y) \qquad (2.6)$$

Here $\zeta^\mu(t, y)$ is a vector-valued function, y is an n-dimensional vector of arbitrary constants. In order to apply the above presented procedure in the general case, it is necessary to introduce additional assumptions. We set $\mu = 1/\nu$, $\nu = 1, 2, 3, \ldots,$ $u^\mu = u^{1/\nu}$.

Assumption 1. The condition

$$\rho\,(u^{1/\nu},\ u^{1/\nu+1}) \to 0, \quad \nu \to \infty$$

(ρ is the distance in an m-dimensional Euclidean space) is satisfied at each point (x, t).

For each point (x, t), by virtue of the closedness and boundedness of set U, from the Bolzano-Weierstrass theorem follows the existence of a subsequence ν_i, $i = 1, 2, \ldots,$ such that $\lim u^{1/\nu_i}$ exists. The uniqueness of this limit

$$u^* = \lim_{i\to\infty} u^{1/\nu_i} \qquad (2.7)$$

follows from Assumption 1.

It is necessary to note further, that ambiguity in the choice of the optimal control occurs, as a rule, not in the whole set of values of (x, t). Thus, in the example considered earlier the control is defined uniquely on the set $|x| = k\tau + 1$. Let Ω_1 be the set of values of (x, τ) for which the optimal control u° is defined uniquely, Ω_2 be the set of those values of (x, τ) for which the choice of the optimal control is not unique.

Assumption 2. The vector u^* defined by equality (2.7) in Ω_2 coincides on the boundary of regions Ω_1 and Ω_2 with the optimal control u° defined uniquely in Ω_1.

As a result we make the following definition.

Definition. Let the conditions of Assumptions 1 and 2 be satisfied. The vector u^* defined by equality (2.7) is called the synthesis of the optimal control for problem

(2.1), (2.2) in the region Ω_2, i.e.

$$u^* = u^\circ, \quad (x, t) \in \Omega_2$$

The equations

$$x = \zeta^\circ(t, y), \ y = (y_1, \ldots, y_n), \ y_i = \text{const}$$

specifying the trajectory field corresponding to control u^* in Ω_2 and u° in Ω_1 are called the optimal trajectory field of problem (2.1), (2.2) and the equations of the characteristic curves of the deterministic Bellman equation (2.3).

We illustrate what we have said by the problem described in Example 1. The function S°, being the solution of the deterministic problem has the form

$$S^\circ(x, \tau) = \begin{cases} 1, & |x| \leqslant k\tau + 1 \\ 0, & |x| > k\tau + 1 \end{cases}$$

Let us construct the function S^μ. According to (2.4) we obtain

$$S_x{}^\mu = (2\sqrt{\pi\tau\mu})^{-1} \exp\left[-\frac{(x + k\tau + 1)^2}{4\mu\tau}\right]\left\{1 - \exp\frac{x(k\tau + 1)}{\mu\tau}\right\}$$

Therefore, the value u^μ, $|u^\mu| \leqslant k$, giving the maximum to the form $u^\mu \ S_x{}^\mu$, is determined by the equality

$$u^\mu = u^\circ = \begin{cases} -k, & x \geqslant 0 \\ k, & x < 0 \end{cases}$$

by definition, the optimal trajectories of the problem, passing through the point (x_0, t_0), have the form

$$x - x_0 = k(\tau - \tau_0), \qquad x_0 \geqslant 0$$

$$x - x_0 = -k(\tau - \tau_0), \qquad x < 0, \ \tau_0 = T - t_0$$

These same curves, by definition, are the characteristics of the deterministic Bellman equation, passing through the point (x_0, τ_0).

3. Construction of the approximate solution. Suppose that we have found the optimal control synthesis $u^\circ(x, t)$ and the optimal trajectory field

$$x = \zeta^\circ(t, y), \ y = (y_1, \ldots, y_n), \ y_i = \text{const} \tag{3.1}$$

of the deterministic problem (2.1), (2.2) and, by the same token, in accordance with Sect. 2 the characteristic curves of the deterministic Bellman equation (2.3) are defined. We assume that the condition

$$\det \| \partial\zeta_j{}^\circ(t, y) \,/\, \partial y_i \| \neq 0$$

is satisfied. Then, solving system (3.1) relative to y, we obtain

$$y = f(x, t), \ y = (y_1, \ldots, y_n), \ f = (f^1, \ldots, f^n)$$

We seek the approximate solution of boundary-value problem (1.3) as a function of the value of the constants y_i such that $y_i = f^i(x, t)$ $(i = 1, \ldots, n)$, and of the values of t. We denote by $S^\circ(y)$ the solution of the deterministic Bellman equation (2.3) in the new variables (y, τ). By virtue of the definition of the characteristic curves of the optimal trajectories of the deterministic problem (2.1), (2.2), these curves possess the property that the function S° retains its constant value along them. Consequently, the discontinuity surface $A^\circ(x, \tau) = 0$ is necessarily the characteristic surface of the deterministic Bellman equation and is one of the optimal trajectories of problem (2.1), (2.2); otherwise the characteristic curves would intersect the surface $A^\circ = 0$ and the function S° would have a jump along these curves, which contradicts the definition of characteristic curves as optimal trajectories of the deterministic problem.

Without loss of generality we can consider that the discontinuity surface $A^\circ = 0$ corresponds to values of the constants $y_i = 0$ $(i = 1, \ldots, m)$, $m \leqslant n$. From what has been said it is clear that the solution of the deterministic problem can be written as

$$S^\circ(y) = \begin{cases} S^{\circ 1}(y), & y_i \leqslant 0 \quad (i = 1, \ldots, m) \\ S^{\circ 2}(y), & y_i > 0 \quad (i = 1, \ldots, m) \end{cases} \tag{3.2}$$

In (1.3) we set the control u equal to the u°-optimal control of the deterministic problem (2.1), (2.2). We denote the corresponding solution of problem (1.3) with control $u = u^\circ$ by $W^\circ(y, \tau)$. The desired function W° satisfies Eq. (1.3) with $u = u^\circ$ if

$$W_\tau^\circ + \sum_{j=1}^{n} W_{y_j}^\circ \frac{\partial y_j}{\partial \tau} = \sum_{k=1}^{m} u_k^\circ \sum_{i=1}^{k} a_{ik} \sum_{j=1}^{n} W_{y_j}^\circ \frac{\partial y_j}{\partial x_i} + \tag{3.3}$$
$$\frac{1}{2}\varepsilon^2 \sum_{i,k=1}^{n} b_{ik}^1 \left[\sum_{j,s=1}^{n} \left(W_{y_j y_s}^\circ \frac{\partial y_j}{\partial x_i}\frac{\partial y_s}{\partial x_k} + W_{y_j}^\circ \frac{\partial^2 y_j}{\partial x_i \partial x_k} \right) \right]$$

From the definition of the characteristics of the deterministic Eq. (1.3) it follows that

$$\frac{\partial y_j}{\partial \tau} = \sum_{k=1}^{m} u_0{}^k \sum_{i=1}^{n} a_{ik} \frac{\partial y_j}{\partial x_i}, \quad j = 1, \ldots, n \tag{3.4}$$

Note. The surface $y = f(x, t)$, $y = (y_1, \ldots, y_n)$, $f = (f^1, \ldots, f^n)$, as simple examples show, can have conic points. The values of the derivative $\partial y_j / \partial x_i$ at the conic point are defined here and subsequently in such a way that identity (3.4) is satisfied.

Let us now consider the group of variables $y_1, \ldots, y_m (m \leqslant n)$ which define the discontinuity surface of function S° in (3.2) and introduce the new variables $z_j = y_j / \varepsilon$, $j = 1, \ldots, m$. The remaining variables $y' = (y_{m+1}, \ldots, y_n)$ remain unchanged. Let us seek the approximate solution of Eq. (3.3) in the form

$$W^\circ = w^\circ + \varepsilon w^1 + O(\varepsilon^2) \tag{3.5}$$

Substituting W°, represented in form (3.5), into Eq. (3.3) and allowing for relations (3.4), we find that the functions w° must satisfy the boundary-value problem

$$w_\tau^\circ = \frac{1}{2} \sum_{j,s=1}^{m} c_{js} w_{z_j z_s}^\circ, \quad w^\circ(z, y', \tau)|_{\tau=0} = \psi(z, y')|_{\tau=0} \tag{3.6}$$
$$c_{js} = \sum_{i,k=1}^{n} b_{ik}^1(\tau) \frac{\partial y_j}{\partial x_i}\frac{\partial y_s}{\partial x_k}$$

Further on we shall assume that c_{js} $(j, s = 1, \ldots, m)$ are functions of the variables $y' = (y_{m+1}, \ldots y_n)$, τ. The function w^1 must be chosen so as to satisfy the equality

$$\omega_\tau^1 = \frac{1}{2} \sum_{j,s=1}^{m} c_{js} w_{z_j z_s}^1 + G(z, y', \tau; w^\circ), w^1(z, y', \tau)|_{\tau=0} = 0 \tag{3.7}$$

Here
$$G(z, y', \tau; w^\circ) = \frac{1}{2}\sum_{j=1}^{m}\sum_{s=m+1}^{n}\sum_{i,k=1}^{n} b_{ik}^1\left(\frac{\partial y_j}{\partial x_i}\frac{\partial y_s}{\partial x_k} \overset{\bullet}{w}_{z_j y_s} + \frac{\partial^2 y_j}{\partial x_i \partial x_k}\overset{\circ}{w}_{z_j}\right)$$

Assume that the condition
$$\sum_{j,s=1}^{m} c_{js}(y', \tau)\lambda_j\lambda_s > 0, \quad \lambda = (\lambda_1, \ldots, \lambda_m) \neq 0$$

is satisfied. Then the original problem reduces to the solving of two boundary-value problems of m th-order parabolic equations; here the number m is determined by that number of variables $y_i (i = 1, \ldots, m)$ necessary to give the discontinuity surface of solution S° of the deterministic Bellman equation.

The solutions of boundary-value problems (3.6), (3.7) are written out explicitly [8]. Let us consider the matrix with elements
$$e_{js} = \int_0^\tau c_{js}(y', \lambda)\, d\lambda$$

By $e^{is}(y', \tau)$ we denote the elements of the matrix inverse to $\| e_{js} \|$ and by $E(y', \tau)$ the determinant of matrix $\| e_{js} \|$. In the variables z, y', τ the solution of the deterministic Bellman equation has the form $S^\circ(y) = S^\circ(\varepsilon z, y')$. Using this, the solution of problem (3.6) can be written as
$$w^\circ = \int S^\circ(\varepsilon\lambda, y')\, p(\lambda - z, y', \tau)\, d\lambda \tag{3.8}$$
$$p(\lambda - z, y', \tau) = (2\pi)^{-m/2} | E |^{-1/2} \exp\left[-\frac{1}{2}\sum_{j,s=1}^{m} e^{js}(\lambda_j - z_j)(\lambda_s - z_s)\right]$$

The integration in (3.8) is carried out over all values of $\lambda = (\lambda_1, \ldots, \lambda_m)$. The solution of boundary-value problem (3.7) is given by the formula
$$w^1 = \int_0^\tau\int G(\lambda, y', \tau^1; w^\circ)\, p(\lambda - z, y', \tau - \tau^1)\, d\lambda d\tau^1$$

Note. The method of constructing the fundamental solution, suggested by Levi [8], is used to solve boundary-value problems (3.6), (3.7) when the coefficients c_{js} are functions of the variables $z = (z_1, \ldots, z_m)$. As a result the boundary-value problem is reduced to solving an integral equation of the second kind, which can be obtained by the method of successive approximations.

The following assertion gives a notion of the structure of the approximate solution $W^{\circ,\varepsilon} = w^\circ + \varepsilon w^1$.

Assertion 1. Let the function $S^\circ(y)$ be thrice continuously differentiable outside the discontinuity surface $y_1 = y_2 = \ldots = y_m = 0$. Then for every $\varepsilon > 0$ and $\tau \in [0, T]$ there exists a neighborhood $\Omega_{\varepsilon,\tau}$ of the discontinuity surface of function S°, outside which the inequalities
$$| w^\circ - S^\circ | \leqslant K\varepsilon, \quad | w_{y_i}^\circ - S_{y_i}^\circ | \leqslant k'\varepsilon \qquad (i = 1, \ldots, m)$$
$$K, K' = \text{const}$$
are valid.

The proof of Assertion 1 relies on the properties of the fundamental solutions of parabolic equations [8, 9].

Corollary. Outside set $\Omega_{\varepsilon,\tau}$, the function S°, being a solution of the deterministic Bellman equation, can be considered as the function w°.

4. Error estimate of the approximate solution $W^{\circ,\varepsilon} = w^\circ + \varepsilon w^1$. Necessary and sufficient conditions for the coincidence of the optimal control synthesis for a system additively containing white noise and that for the corresponding deterministic system. By $u^{1,\varepsilon} \in U$ we denote the control which gives the minimum to the form

$$\min_{u \in U} \left\{ \sum_{k=1}^{m} u_k \sum_{i=1}^{n} a_{ik} W^{0,\varepsilon}_{x_i} \right\} = \sum_{k=1}^{m} u_k^{1,\varepsilon} \sum_{i=1}^{n} a_{ik} W^{0,\varepsilon}_{x_i} \tag{4.1}$$

Here $W^{\circ,\varepsilon} = w^\circ + \varepsilon w^1$ is an approximate solution of the Bellman equation (1.3) with control $u = u^\circ$ corresponding to the optimal deterministic problem. We introduce the notation

$$H(u, \cdot) = \sum_{k=1}^{m} u_k \sum_{i=1}^{n} a_{ik} \frac{\partial}{\partial x_i} (\cdot)$$

$$L(\cdot) = \frac{1}{2} \varepsilon^2 \sum_{i,k=1}^{n} b_{ik}^1 \frac{\partial^2 (\cdot)}{\partial x_i \partial x_k} - \frac{\partial}{\partial \tau} (\cdot)$$

The following result is valid.

Theorem 2. Let the twice-differentiable bounded function $W^{\circ,\varepsilon}$ satisfy the equation

$$H(u^\circ, W^{\circ,\varepsilon}) + L(W^{\circ,\varepsilon}) = \alpha^\varepsilon (x, \tau), \quad W^{\circ,\varepsilon}(x, 0) = \psi(x) \tag{4.2}$$

$$(k^\varepsilon \leqslant \alpha^\varepsilon (x, \tau) \leqslant K^\varepsilon, \ k^\varepsilon, \ K^\varepsilon = \text{const})$$

and let

$$u^{1,\varepsilon} - u^\circ = \beta^\varepsilon (x, \tau), \quad \beta^\varepsilon = (\beta_1^\varepsilon, \ldots, \beta_m^\varepsilon) \tag{4.3}$$

($u^{1,\varepsilon}$ is the control defined by equality (4.1), u° is the optimal control for the deterministic problem (2.1), (2.2). Then, the inequalities

$$H(\beta^\varepsilon, W^{\circ,\varepsilon}) \leqslant 0 \tag{4.4}$$

$$\tau (k^\varepsilon + C^\varepsilon) \leqslant S - W^{\circ,\varepsilon} \leqslant \tau K^\varepsilon \tag{4.5}$$

are valid. Here S is a solution of the Bellman equation (1.3), C^ε is a constant such that

$$H(\beta^\varepsilon, W^{\circ,\varepsilon}) \geqslant C^\varepsilon$$

Proof. Let us first prove inequalitv (4.4). Since $H(u^{1,\varepsilon}, W^{\circ,\varepsilon}) \leqslant H(u^\circ, W^{\circ,\varepsilon})$, because form H is linear in u we obtain

$$H(u^{1,\varepsilon} - u^\circ, \ W^{\circ,\varepsilon}) = H(\beta^\varepsilon, \ W^{\circ,\varepsilon}) \leqslant 0$$

We write Eq. (3.1) as

$$H(u^*, S) + L(S) = 0, \quad S(x, 0) = \psi(x)$$

Here u^* is the optimal control for problem (1.1). Since $H(u^*, S) \leqslant H(u^\circ, S)$, the inequality $H(u^\circ, S) + L(S) \geqslant 0$ is satisfied. Subtracting this inequality from (4.2), we have

$$H(u^\circ, W^{\circ,\varepsilon} - S) + L(W^{\circ,\varepsilon} - S) \leqslant \alpha^\varepsilon (x, \tau), (W^{\circ,\varepsilon} - S) \mid_{\tau=0} = 0$$

Consider the function $Z = W^{0,\varepsilon} - S + K^{\varepsilon}\tau$. The inequality

$$H(u^{\circ}, Z) + L(Z) \leqslant \alpha^{\varepsilon}(x, \tau) - K^{\varepsilon} \leqslant 0, \quad |Z|_{\tau=0} = 0$$

is valid for it. Using the maximum principle for parabolic equations in an unbounded region [9], we obtain $Z \geqslant 0$. Hence it follows that

$$S - W^{\circ,\varepsilon} \leqslant K^{\varepsilon}\tau \tag{4.6}$$

On the other hand,

$$H(u^*, W^{\circ,\varepsilon}) \geqslant H(u^{1,\varepsilon}, W^{\circ,\varepsilon})$$

Here $u^{1,\varepsilon}$ is the control defined by equality (4.1). From equality (4.3) follows the inequality

$$H(u^*, W^{\circ,\varepsilon}) - H(\beta^{\varepsilon}, W^{\circ,\varepsilon}) \geqslant H(u^{\circ}, W^{\circ,\varepsilon})$$

Therefore, from (4.2) we have

$$H(u^*, W^{\circ,\varepsilon}) + L(W^{\circ,\varepsilon}) \geqslant \alpha^{\varepsilon}(x, \tau) + H(\beta^{\varepsilon}, W^{\circ,\varepsilon})$$

The inequality

$$H(u^*, Z^1) + L(Z^1) \leqslant k^{\varepsilon} + C^{\varepsilon} - H(\beta^{\varepsilon}, W^{\circ,\varepsilon}) - \alpha^{\varepsilon}(x, \tau) \leqslant 0$$

$$Z^1|_{\tau=0} = 0, \; Z^1 = S - W^{\circ,\varepsilon} - (k^{\varepsilon} + C^{\varepsilon})\tau$$

is valid. Applying once more the maximum principle for parabolic equations, we obtain

$$S - W^{\circ,\varepsilon} \geqslant (k^{\varepsilon} + C^{\varepsilon})\tau$$

Hence from (4.6) follows the required estimate (4.5).

Note. The result of Theorem 2 remains valid for systems of more general form

$$dx/dt = a(x, t)\, u + c(x, t) + b(x, t)\,\xi$$

Here it is assumed that the elements of matrices a, b and c are bounded functions. The final-value functional J can grow as $|x| \to \infty$ no faster than the function $\exp[d(r^2 + 1)]$, where d is some positive constant $r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$. The latter conditions are required by the Theorem 9 of [9], on which the proof of Theorem 2 is based.

Corollary 1. Let the vector-valued function $\beta^{\varepsilon}(x, \tau)$ be such that $0 \geqslant H(\beta^{\varepsilon}, W^{\circ,\varepsilon}) \geqslant C\varepsilon^2, C = \text{const}$. Then the estimate

$$|S - W^{\circ,\varepsilon}| \leqslant K'\varepsilon^2, \quad K' = \text{const} \tag{4.7}$$

is valid.

In fact, by construction the function $W^{\circ,\varepsilon}$ satisfies Eq. (3.3) to within terms of order $O(\varepsilon^2)$; therefore, $|\alpha^{\varepsilon}(x, \tau)| \leqslant K\varepsilon^2$, $K = \text{const}$. Then estimate (4.7) is satisfied with a constant $K' = (K + |C|)\,T$.

Corollary 2. (Necessary and sufficient conditions for the coincidence of the optimal control synthesis for systems additively containing Gaussian white noise and that for the corresponding deterministic systems). The optimal control u° for the deterministic problem (2.1) with functional (1.2) coincides identically with the optimal control of the perturbed problem (1.1) with the same functional (1.2) if

$$u^{\circ} = u^1 \tag{4.8}$$

Here $u^1 \in U$ is the control giving a minimum to the form

$$\min_{u \in U} H(u, W^{\circ}) = H(u^1, W^{\circ}) \tag{4.9}$$

W° is a solution of the boundary-value problem (3.3)

$$H(u^{\circ}, W^{\circ}) + L(W^{\circ}) = 0, \ W^{\circ}(x, 0) = \psi(x)$$

u° is the optimal control for the deterministic problem (2.1).

Proof. Necessity. Let u^* be the optimal control for the perturbed problem (1.1) with functional (1.2), such that $u^* = u^{\circ}$. Then the Bellman function S being a solution of boundary-value problem (1.3), identically equals W°, being a solution of problem (3.3). Consequently, the control u^1 defined by equality (4.9) is such that $u^1 = u^*$. Hence it follows that $u^{\circ} = u^1$.

Sufficiency. Let (4.8) be valid. Applying the result of Theorem 2 with $\alpha^{\varepsilon} = 0$, $\beta^{\varepsilon} = 0$, we obtain that $S = W^{\circ}$. Here S is a solution of boundary-value problem (1.3). Therefore, the control u^1 defined by equality (4.9) coincides identically with the optimal control for problem (1.1), i.e. $u^1 = u^*$. Consequently, by virtue of (4.8), $u^* = u^{\circ}$. Q.E.D.

Corollary 3. Let $u^{\circ} \neq u^1$. We consider a solution of the Bellman equation (1.3) with control u^1, i.e.

$$H(u^1, W^1) + L(W^1) = 0, \ W^1(x, 0) = \psi(x)$$

Further, from function W^1 we find in the same way as in (4.9) the control u^2 etc. [10]. If at some κth step $u^k = u^{k+1}$, then the last equality is the necessary and sufficient condition for $u^* = u^k$. Here u^* is the optimal control for the original problem (1.1).

Example 2. Consider the system

$$d^2x/dt^2 = u + \varepsilon\zeta, \quad x = (x_1, x_2), \quad t \in [0, T]$$
$$u = (u_1, u_2), \quad |u_1| \leqslant k_1, \quad |u_2| \leqslant k_2$$

($\zeta = (\zeta_1, \zeta_2)$ is a vector of independent Gaussian white noises of unit intensity). We are required to construct a control method maximizing the probability of hitting onto the set $|x_1| + |x_2| \leqslant 1$ at the instant $t = T$. We introduce the new variables $y_k = (T - t)\,x_k + x_k$, $k = 1, 2$. Note that $x_k(T) = y_k(T)$, $k = 1, 2$. The original equations take the form

$$dy/dt = (T - t)(u + \varepsilon\zeta), \quad y = (y_1, y_2), \quad u = (u_1, u_2), \quad \zeta = (\zeta_1, \zeta_2)$$

The Bellman function of the deterministic problem is a simple indicator of the attainability set, taking the values

$$S^{\circ} = \begin{cases} 1, & \| y_1 | - k_1\tau^2/2 | + | | y_2 | - k_2\tau^2/2 | \leqslant 1 \\ 0 & \text{in the remaining cases} \end{cases}$$

We construct the function S^{μ} according to (2.4). From the results of Sect. 2 it follows that the optimal control for the deterministic problem is defined by the formula

$$u_i^{\circ} = \begin{cases} -k_i, & y_i \geqslant 0 \\ k_i, & y_i < 0 \end{cases} \quad (i = 1, 2)$$

The optimal trajectories, being the characteristics of the deterministic Bellman equation, have the form

$$\eta_i = |y_i| - k_i\tau^2/2 \quad (i = 1, 2)$$

We introduce the new variables $z_i = \eta_i/\varepsilon$, τ $(i = 1, 2)$.

According to (3.6) the function w° satisfies the boundary-value problem

$$w_\tau^\circ = \frac{1}{2}\tau^2\Delta w^\circ, \quad w^\circ\Big|_{\tau=0} = \begin{cases} 1, & |z_1|+|z_2| \leqslant \varepsilon^{-1} \\ 0, & |z_1|+|z_2| > \varepsilon^{-1} \end{cases}$$

As follows from (3.7) the function w^1 is such that

$$w_\tau^1 = \frac{1}{2}\tau^2\Delta w^1, \quad w^1\Big|_{\tau=0} = 0, \quad \Delta = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}.$$

From the uniqueness of the Cauchy problem for the heat conduction equation it follows that $w^1 = 0$. We note that the derivatives of the functions η_1, η_2 with respect to the variables y_1, y_2 at the points $y_1 = 0$, $y_2 = 0$ are to be understood in the sense indicated in Sect. 3. According to (3.8) the function w° has the form

$$w^\circ = (\pi\tau^3\varepsilon^2)^{-1} \iint\limits_{|\lambda_1|+|\lambda_2|\leqslant 1} \exp\left\{-\frac{1}{2\varepsilon^2}\left[\frac{(\lambda_2-\eta_1)^2+(\eta_2-\lambda_2)^2}{\tau^6}\right]\right\} d\lambda_1 d\lambda_2$$

Direct verification shows that the control u^1 defined by formula (4.1) is such that assumption (4.8) is satisfied, namely, $u^\circ = u^1$. From Theorem 2 it follows that the latter formula yields an approximate solution of the Bellman equation of the original problem to within $O(\varepsilon^2)$.

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